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SIMULATION OF A VISCOUS COMPRESSIBLE MULTICONSTITUENT FLUID WITH ALLOWANCE FOR POLARIZATION AND MAGNETIZATION EFFECTS

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A variational equation is used for deriving a closed system of equations for defining the behavior of a viscous compressible multiconstituent fluid [1]. The determining parameters comprise besides density, entropy, and mass concentration of constituents, also the polarization and magnetization vectors of individual constituents. In conformity with the method developed in [2 – 4] the mixture is considered to be a single continuous medium so that the presence of various constituents results in additional degrees of internal freedom in the definition of the considered medium. Chemical reactions between mixture constituents and phase transitions are assumed to be absent (*).

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*) The simulation of a viscous multiconstituent fluid with allowance for diffusion and

(continued on the next page)

1. Basic assumptions and determining parameters of simulation.

Let us consider the simulation of a medium consisting of N material continua which we shall call mixture constituents which fill one and the same volume of a three-dimensional Euclidean space. We assume that a separate system of coordinates frozen into the medium is attached to each mixture constituent. Points in that system of coordinates will be denoted by ξ_i^β (here and subsequently Greek letter superscripts run through the values 1, 2, 3, while Latin subscripts relate marked quantities to particular mixture constituents and run through the values 0, 1, 2, 3,..., $N-1$). In the observer's system of coordinates with basis vectors $\ni_\alpha$ and covariant components of the metric tensor denoted by $g_{\alpha\beta}$ the coordinates of a point will be denoted by x^α. The law of motion of the i-th mixture constituent is defined by functions $x^\alpha = x^\alpha(\xi_i^\beta, t)$, where t is the absolute time. We define components of the velocity vector v_i^α and the density ρ_i of the i-th constituent by equalities

$$v_i^\alpha = \left.\frac{\partial x^\alpha(\xi_i^\beta, t)}{\partial t}\right|_{\xi_i^\beta=\text{const}} \tag{1.1}$$

$$\rho_i = \frac{\omega_i(\xi_i^\beta)}{\sqrt{g_i^\wedge}}, \qquad g_i^\wedge = \det\|g_{i\hat{\alpha\beta}}\|$$

where $g_{i\hat{\alpha\beta}}$ are components of the metric tensor in the system of coordinates attached to the i-th mixture constituent. Chemical reactions and phase transitions between constituents are assumed absent, since otherwise it would have been impossible to consider parameters ω_i as functions of only the attached coordinates.

The density ρ and velocity vectors v^α of the mixture as a whole are determined by formulas

$$\rho = \sum_{i=0}^{N-1} \rho_i, \qquad v^\alpha = \sum_{i=0}^{N-1} c_i v_i^\alpha \tag{1.2}$$

where $c_i = \rho_i / \rho$ is the mass concentration of the i-th constituent.

Below we also use a system of coordinates attached to the mixture as a whole, which can be introduced by the differential relationships

$$v^\alpha = \left.\frac{\partial x^\alpha}{\partial t}\right|_{\xi^\beta=\text{const}} = \sum_{i=0}^{N-1} c_i \left.\frac{\partial x^\alpha}{\partial t}\right|_{\xi_i^\beta=\text{const}} \tag{1.3}$$

where the constants of integration ξ^β are by definition the coordinates attached to the mixture as a whole.

The mass density of the mixture as a whole can now be represented in the form $\rho = f(\xi^\beta) g^{\wedge -1/2}$, $g^\wedge = \det //g_{\hat{\alpha\beta}}//$, where $g_{\hat{\alpha\beta}}$ are metric tensor components related to the system of coordinates attached to the mixture as a whole, and $f(\xi^\beta)$ is a function of only the coordinates attached to the latter.

Taking into consideration the definition (1.2) of the mixture velocity and (1.1), for the relationship between the displacement $d_i x^\alpha$ of the i-th constituent and dx^α of the mixture as a whole considered at one and the same point at coordinates x^α in the Eulerian space at one and the same instant of time t, we obtain

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chemical reactions with the use of the basic variational equation was considered in detail by the authors in "Simulation of continuous media in Newtonian mechanics", Report NII of Mechanics, MGU, №1468, 1973.

$$dx^\alpha = \sum_{i=0}^{N-1} c_i d_i x^\alpha \tag{1.4}$$

Formula (1.4) and the obvious equality

$$d_i x^\alpha = dx^\alpha + (v_i^\alpha - v^\alpha)\, dt$$

imply that equality

$$d_i\chi\,(x^\alpha,\ t) = d\chi + \nabla_\alpha\chi\,(v_i^\alpha - v^\alpha)\, dt$$

is valid for the differential of any tensor function with components $\chi\,(x^\alpha, t)$ determined in the observer's system of coordinates and considered at one and the same point of the Eulerian space at one and the same instant of time t.

In simulating a mixture consisting of N constituents, each of which may be polarized and magnetized to various degrees, we define the properties of such processes (which in the following are considered reversible) by the three-dimensional vectors of polarization $\mathbf{P}_i{}^*$ and magnetization $\mathbf{M}_i{}^*$ related to a unit volume of the i-th constituent. These vectors are introduced in their own inertial system of coordinates related to the i-th constituent with contravariant components $P_i{}^{*\alpha}$ and $M_i{}^{*\alpha}$ (here and subsequently the term "own inertial system of coordinates" is understood in the meaning given to it in [1]). Vectors and tensors (as well as their components) are denoted in this system by asterisks. We further assume that one of the mixture constituents may carry a free electric charge of density ρ_{e0} and that conduction currents defined by the volume density vector of the electric conduction current i_0 may flow in that constituent. Note that the continuity equation which in the observer's system of coordinates is of the form

$$\frac{\partial\rho_{e0}}{\partial t} + \nabla_\alpha\,(i_0{}^\alpha + \rho_{e0}v^\alpha) = 0 \tag{1.5}$$

where ∇_α is the operator of covariant differentiation in the observer's system of coordinates (it is assumed that in what follows summation is carried out with respect to concurrent upper and lower Greek letter sub-indices). In accordance with our assumptions, we set $\rho_{ek},\ i_k = 0\ (k \neq 0)$ for the remaining components. We assume that velocities of the mixture as a whole and of its individual constituents are reasonably low in comparison with the speed of light in vacuum, hence terms of the order of $v\,/\,c$ and $v_i\,/\,c$ are neglected.

The electric intensity vector $\mathbf{E}$ and the magnetic induction vector $\mathbf{B}$ in the observer's system of coordinates with covariant components E^α and B^α, respectively, are taken as the characteristics of the electromagnetic field in the mixture as a whole. We also use the covariant components of vectors $\mathbf{P}_i{}^*$ and $\mathbf{M}_i{}^*$ determined in the observer's system of coordinates and denoted by $P_i{}^\alpha$ and $M_i{}^\alpha$. These components are related to the covariant components $P_i{}^{*\alpha}$ and $M_i{}^{*\alpha}$ by the usual formulas for vector component transformation by virtue of the assumption of smallness of terms of the order of $v_i\,/\,c$.

Using the vector and scalar potentials of the electromagnetic field in the medium, we obtain for the components of vectors E^α and B^α the equalities

$$E_\alpha = -\frac{1}{c}\frac{\partial A_\alpha}{\partial t} - \nabla_\alpha\varphi, \qquad B^\alpha = \varepsilon^{\alpha\beta\gamma}\nabla_\beta A_\gamma \tag{1.6}$$

where $\varepsilon^{\alpha\beta\gamma}$ are contravariant components of the Levi-Civita tensor which is antisymmetric for all indices. It follows from (1.6) that vectors $\mathbf{E}$ and $\mathbf{B}$ satisfy the second pair of Maxwell's equations which have to be added to the derived below system of equations

$$\varepsilon^{\alpha\beta\gamma}\nabla_\beta E_\gamma = -\frac{1}{c}\frac{\partial B^\alpha}{\partial t}, \qquad \nabla_\alpha B^\alpha = 0 \tag{1.7}$$

2. The variational equation. Let us introduce variations of functions

$$x^\alpha(\xi^\mu,\ t),\ A_\alpha(\xi^\mu,\ t),\ \varphi(\xi^\mu,\ t),\ S(\xi^\mu,\ t),\ \xi_i{}^\alpha(\xi^\mu,\ t)$$

which we denote by δx^α, δA_α, $\delta\varphi$, δS and $\delta\xi_i{}^\alpha$ (δ denotes variations of quantities which are functions of coordinates attached to the mixture as a whole and of time). For any function $\psi(\xi^\mu,\ t)$ the following definition of variation is justified:

$$\delta\psi(\xi^\mu,\ t) = \psi'(\xi^\mu,\ t) - \psi(\xi^\mu,\ t) \tag{2.1}$$

where the prime denotes the value of the varied function. Let us determine the variation of functions related to the system of coordinates attached to the i-th constituent $x^\alpha(\xi_i{}^\mu,\ t)$, $\hat{P}_i{}^\alpha(\xi_i{}^\mu,\ t)$ and $\hat{M}_i{}^\alpha(\xi_i{}^\mu,\ t)$ by the equalities

$$\begin{aligned}
\delta_i x^\alpha &= x^{\alpha\prime}(\xi_i{}^\mu, t) - x^\alpha(\xi_i{}^\mu, t) \\
\delta_i \hat{P}_i{}^\alpha &= \hat{P}_i{}^{\alpha\prime}(\xi_i{}^\mu, t) - \hat{P}_i{}^\alpha(\xi_i{}^\mu, t) \\
\delta_i \hat{M}_i{}^\alpha &= \hat{M}_i{}^{\alpha\prime}(\xi_i{}^\mu, t) - \hat{M}_i{}^\alpha(\xi_i{}^\mu, t)
\end{aligned} \tag{2.2}$$

Definitions of the differently defined variations of coordinates (see formulas (2.1) and (2.2)) imply the following dependence between these:

$$\delta_i x^\alpha = \delta x^\alpha - \delta\eta_i{}^\alpha \qquad \left(\delta\eta_i{}^\alpha = \frac{\partial x^\alpha}{\partial \xi_i{}^\mu}\,\delta\xi_i{}^\mu\right)$$

while variations $\delta\xi_i{}^\alpha$ are bound by the relationships

$$\sum_{i=0}^{N-1} c_i \delta\eta_i{}^\alpha = 0$$

With the use of these variations of the above functions it is not difficult to derive variations of functions $E^\alpha(\xi^\mu,\ t)$ and $B^\alpha(\xi^\mu,\ t)$ (with allowance for relationships (1.6)) and, also, variations of densities $\rho_i(\xi^\mu,\ t)$, of concentration $c_i(\xi^\mu,\ t)$ and of components of velocity vectors $v^\alpha(\xi^\mu,\ t)$ and $v_i{}^\alpha(\xi^\mu,\ t)$.

As the basis for the simulation of a mixture consisting of N constituents, we take a variational equation of the form

$$\delta\int\limits_{V_4} \Lambda\, d\tau_4 + \delta W^* + \delta W = 0, \qquad \Lambda = \Lambda_1 + \Lambda_2 \tag{2.3}$$

where Λ is the combined Lagrangian of the electromagnetic field and medium, δW^* is some specified functional whose method of selection and form are defined below, δW is a functional determined by the specified Λ and δW^* along the three-dimensional boundary Σ_3 of an arbitrary four-dimensional volume V_4. We define the Lagrangian Λ_1 of the electromagnetic field in the medium by the expression

$$\Lambda_1 = \frac{1}{8\pi}(E^\alpha E^\beta g_{\alpha\beta} - B^\alpha B^\beta g_{\alpha\beta}) + E^\alpha \sum_{i=0}^{N-1} P_i{}^\beta g_{\alpha\beta} + B^\alpha \sum_{i=0}^{N-1} M_i{}^\beta g_{\alpha\beta}$$

The Lagrangian Λ_2 of the medium is assumed to be equal $\rho(K - U)$, where U is the internal energy of a unit mass of the medium and K is the kinetic energy of such unit mass, equal to the sum of kinetic energies of macroscopic motions of individual constituents

$$K = \sum_{i=0}^{N-1} \frac{c_i v_i^2}{2}$$

The selected definition of kinetic energy — which shows its additivity — is in many instances either insufficient or can be simplified (e. g. in many instances it is possible to take $v^2 / 2$ as the kinetic energy of a unit of mass). The results presented here can be readily extended to cases of any arbitrary dependence of the Lagrangian Λ_2 on related arguments (thus, for example, small-scale motions of individual constituents can be taken into account). Note that the definition of kinetic energy used here yields the most rational form of equations for the momenta of individual constituents.

In what follows we assume that the density of internal energy of the mixture considered as a whole is a function of densities ρ_i of individual constituents, of entropy S, and of contravariant components of polarization π_i^α and magnetization m_i^α per unit of mass of the i-th constituent. (Components of vectors π_i^α and m_i^α are related to components of vectors P_i^α and M_i^α by equalities $P_i^\alpha = \rho_i \pi_i^\alpha$ and $M_i^\alpha = \rho_i m_i^\alpha$, respectively). In addition to the arguments indicated above, the function of internal energy density of the medium depends on components of the metric tensor $g_{\alpha\beta}$ in the observer's system of coordinates and on certain constants K_B which define physical properties of the mixture (instead of ρ_i parameters c_i and ρ may be taken as the arguments for internal energy).

The form of dependence of the function of internal energy on the indicated set of arguments is not fixed here. It can be selected in a quadratic form of the polarization and magnetization vector components which may be expressed in terms of total (for the mixture as a whole) polarization and magnetization vectors only. The coefficients of such forms can be functions of constituent densities (or of the density of the mixture as a whole and of concentration c_i). As indicated previously, the entropy S calculated per unit of mass of the mixture as a whole appears as one of the parameters defining the internal energy density. The part of that entropy apportioned to the i-th constituent is not considered as a parameter in our simulation. This is because in it we are exclusively concerned with temperature T and consider that at one and the same point of space and at the same instant of time the temperature T_i of individual constituents is equal to the temperature of the mixture as a whole.

We bear in mind the reversibility of the polarization and magnetization processes, but assume that mixture constituents may possess viscosity properties. Assuming that the energy exchange between the electromagnetic field and the mixture is determined by the release of Joule heat in the mixture and by the polarization and magnetization processes, we can express the second law of thermodynamics for the mixture as a whole in the form

$$\rho T dS = \sum_{i=0}^{N-1} \tau_i^{\alpha\beta} \nabla_\beta v_{i\alpha} dt - \sum_{i=0}^{N-1} G_{i\alpha} J_i^\alpha dt + E_\alpha i_0^\alpha dt + dQ_0 \qquad (2.4)$$

$$J_i^\alpha = \rho_i (v_i^\alpha - v^\alpha)$$

where the first term in the right-hand part represents the work of viscosity stresses along displacement gradient of constituents, while the second term represents the work of generalized internal mass forces G_i along the relative displacements of constituents, which (together with generalized forces $\tau_i^{\alpha\beta}$) determine the irreversibility of processes taking place in the medium. J_i^α denote components of the constituent diffusion vectors and $dq_0 = \rho^{-1}dQ_0$ is the input of external heat energy to a unit of mass (of mixture) as a whole, which is unrelated to energy transfer by diffusion and to the action of the electromagnetic field on the medium.

In the region of continuous motion we include in the functional δW^* the following:

1) virtual work $\delta A^{(e)}$ of volume forces with components $F_{i\alpha}$, external relative to the system medium-electromagnetic field, along virtual displacements $\delta_i x^\alpha$ of mixture constituents;

2) possible input of external energy to the medium (unrelated to the addition of energy owing to diffusion processes and to the release of Joule heat), because of the assumption that the second law of thermodynamics (2.4) is satisfied for virtual displacements and possible variation of the medium properties;

3) terms which define the increase of uncompensated heat dQ' owing to dissipative effects related to the release of Joule heat.

Taking the above into consideration, we specify the functional by the following formula:

$$\delta W^* = \int\limits_{V_4} \Big\{\rho T\delta S - \sum_{i=0}^{N-1} \tau_{i\alpha}{}^\beta \nabla_\beta \delta_i x^\alpha - \sum_{i=0}^{N-1} (F_{i\alpha} + \rho_i G_{i\alpha})\, \delta\eta_i{}^\alpha + \qquad (2.5)$$

$$\sum_{i=0}^{N-1} F_{i\alpha}\delta x^\alpha + \frac{1}{c}\, j_0{}^\alpha \delta_{0L} A_\alpha - \rho_{e0}\delta_{0L}\varphi\Big\}\, d\tau_4$$

where the terms defined above are of the form obtained by transformation by formulas (2.2), and $\delta_{0L}A_\alpha$ and $\delta_{0L}\varphi$ denote absolute variations of the vector and scalar potentials for which relationships

$$\delta_{0L}A_\alpha = \delta A_\alpha - \nabla_\beta A_\alpha \delta\eta_0{}^\beta + A_\beta \nabla_\alpha \delta_0 x^\beta$$

$$\delta_{0L}\varphi = \delta\varphi - \nabla_\beta \varphi \delta\eta_0{}^\beta - \frac{1}{c} A_\beta \frac{\partial \delta_0 x^\beta}{\partial t}$$

are valid. With the substitution of actual increments for variations the absolute increments of vector components acquire the meaning of vector component increments of the related system of coordinates attached to the zero mixture constituent, and $j_0{}^\alpha$ are components of the total current of the latter constituent in the observer's system of coordinates. These components are related to the conduction current and the free charge of the zero constituent by equalities $j_0{}^\alpha = i_0{}^\alpha + \rho_{e0} v_0{}^\alpha$

3. The system of equations which defines the mixture simulation. With allowance for the above assumptions about the form of density functions of the internal energy U of the mixture as a whole and about the density of the mixture kinetic energy K, as well as about the form of functional δW^* owing to the assumed arbitrariness of some of the systems of variations, it is possible to derive from the variational equation (2.3) closed systems of equations for determining the simulation of a mixture consisting of N constituents, each of which may be polarized and magnetized to various degrees. As one of possible systems of independent variations we select variations whose arbitrariness we assume to be

$$\delta A_\alpha, \quad \delta\varphi, \quad \delta_i x^\alpha, \quad \delta S, \quad \delta_i \pi_i{}^{\hat{}\alpha}, \quad \delta_i m_i{}^{\hat{}\alpha} \qquad (3.1)$$

This choice of the system of independent variations and the condition that the volume integral of the variational equation is zero, yields the following system of Euler's equations:

$$\varepsilon^{\alpha\beta\gamma}\nabla_\beta H_\gamma = \frac{1}{c}\frac{\partial D^\alpha}{\partial t} + \frac{4\pi}{c} j_0{}^\alpha, \quad \nabla_\alpha D^\alpha = 4\pi\rho_{e0} \qquad (3.2)$$

which represents the first pair of Maxwell's equations defined in the observer's system of coordinates, and where D^α and H_γ are determined by the equalities

$$D^\alpha = E^\alpha + 4\pi \sum_{i=0}^{N-1} P_i{}^\alpha, \quad H_\gamma = B_\gamma - 4\pi \sum_{i=0}^{N-1} M_{i\gamma}$$

$$\rho_i a_{i\alpha} = \nabla_\beta p_{i\alpha}{}^\beta + F_{i\alpha} + F_{Mi\alpha} + \rho_i \sum_{j=0}^{N-1} R_{j\alpha} - \rho R_{i\alpha} + \rho_i G_{i\alpha}. \qquad (3.3)$$

which are the equations of momenta for the i-th constituent of the mixture. We denote by $a_{i\alpha}$ the covariant vector components of acceleration of the i-th constituent of the mixture, which are related to velocity vector components of that constituent by equalities

$$a_{i\alpha} = \frac{\partial v_{i\alpha}}{\partial t} + v_i{}^\gamma \nabla_\gamma v_{i\alpha}$$

Tensor components $p_{i\alpha}{}^\beta$ and $R_{i\alpha}$ are determined by formulas

$$p_{i\alpha}{}^\beta = -\rho\rho_i \frac{\partial U}{\partial\rho_i}\delta_\alpha{}^\beta + \rho\frac{\partial U}{\partial\pi_i{}^\alpha}\pi_i{}^\beta + \rho\frac{\partial U}{\partial m_i{}^\alpha} m_i{}^\beta + \frac{1}{2}(E_\gamma P_i{}^\gamma +$$

$$B_\gamma M_i{}^\gamma)\delta_\alpha{}^\beta - E_\alpha P_i{}^\beta - B_\alpha M_i{}^\beta + \tau_{i\alpha}{}^\beta$$

$$R_{i\alpha} = \frac{\partial U}{\partial\rho_i}\nabla_\alpha\rho_i + \frac{\partial U}{\partial\pi_i{}^\beta}\nabla_\alpha\pi_i{}^\beta + \frac{\partial U}{\partial m_i{}^\beta}\nabla_\alpha m_i{}^\beta$$

The specific form of expressions for components $p_{i\alpha}{}^\beta$ and $R_{i\alpha}$ depends on the choice of arguments of the internal energy of the mixture as a whole and of the set of independent variables. Separation of the total force acting on the i-th constituent into volume and surface components generally requires additional assumptions. We denote by $F_{Mi\alpha}$ the covariant components of the vector of the three-dimensional volume ponderomotive force (which conforms to the Minkowski hypothesis about the form of the four-dimensional tensor of energy-momentum of an electromagnetic field in a medium). For $F_{Mi\alpha}$ we have the following expressions:

$$F_{Mi\alpha} = \rho_{ei}E_\alpha + \frac{1}{c}\varepsilon_{\alpha\beta\gamma} j_i{}^\beta B^\gamma + \qquad (3.4)$$

$$\frac{1}{2}(P_i{}^\gamma\nabla_\alpha E_\gamma - E_\gamma\nabla_\alpha P_i{}^\gamma + M_i{}^\gamma\nabla_\alpha B_\gamma - B_\gamma\nabla_\alpha M_i{}^\gamma)$$

The equations of state, which represent the definition of temperature, are

$$T = \partial U/\partial S \qquad (3.5)$$

The relationship between vectors of electric field intensity and those of magnetic induction in a medium is determined by the equation of state which in terms of projections on coordinates of the observer's system are of the form

$$c_i E_\alpha = \frac{\partial U}{\partial \pi_i{}^\alpha}, \qquad c_i B_\alpha = \frac{\partial U}{\partial m_i{}^\alpha} \tag{3.6}$$

From the variational equation (2.3) we obtain, besides Euler's equations, expressions for the functional δW which may be represented in the form

$$\delta W = \int\limits_{\Sigma_3} \Big\{ \sum_{i=0}^{N-1} (P_{i\alpha}^{\beta} N_\beta + P_{i\alpha}^{4} N_4)\, \delta_i x^\alpha - S_\alpha{}^\beta N_\beta \delta x^\alpha - S_\alpha{}^4 N_4 \delta x^\alpha + \tag{3.7}$$
$$\left[\frac{1}{c} j_0{}^\alpha A_\beta N_\alpha + \rho_{e0} A_\beta N_4\right] \delta\eta_0{}^\beta - \sum_{i=0}^{N-1} \rho_i \frac{v_i{}^2}{2} N_\beta \delta\eta_i{}^\beta -$$
$$\frac{1}{4\pi} \varepsilon^{\alpha\beta\gamma} H_\gamma N_\beta \delta_L A_\alpha + \frac{1}{4\pi c} D^\alpha N_4 \delta_L A_\alpha + \frac{1}{4\pi} D^\alpha N_\alpha \delta_L \varphi \Big\}\, d\sigma_3$$

where $\delta_L A_\alpha$ and $\delta_L\varphi$ denote absolute variations of vector and scalar components of the electromagnetic field in the medium relative to the system of coordinates attached to the mixture as a whole and associated with ordinary variations (in the observer's system of coordinates) by formulas

$$\delta_L A_\alpha = \delta A_\alpha + A_\beta \nabla_\alpha \delta x^\beta$$
$$\delta_L \varphi = \delta\varphi - \frac{1}{c} A_\beta \frac{\partial \delta x^\beta}{\partial t}$$

while for the remaining tensor quantities in the formula for the functional δW we have expressions

$$P_{i\alpha}{}^\beta = p_{i\alpha}{}^\beta - \rho_i v_{i\alpha} v_i{}^\beta$$
$$P_{i\alpha}{}^4 = -\rho_i v_{i\alpha}$$
$$S_\alpha{}^\beta = -\frac{1}{4\pi}(E_\alpha D^\beta + H_\alpha B^\beta) + \frac{1}{8\pi}(E_\gamma D^\gamma + H_\gamma B^\gamma)\,\delta_\alpha{}^\beta$$
$$S_\alpha{}^4 = \frac{1}{4\pi c} \varepsilon_{\alpha\beta\gamma} D^\beta B^\gamma$$

where in conformity with the Minkowski hypothesis $S_\alpha{}^\beta$ denotes of the spatial part of the energy-momentum tensor of the electromagnetic field in the medium and $S_\alpha{}^4$ denotes momentum density components of the electromagnetic field in the medium (the Minkowski hypothesis about the form of the four-dimensional energy-momentum tensor of the electromagnetic field in a medium). N_α and N_4 are components of the vector of the four-dimensional normal to the three-dimensional surface Σ_3 which bounds volume V_4.

4. The equation of energy for a mixture considered as a single medium. As a corollary of the variational equation (2.3) and expressions for the functionals δW^* and δW, from formulas (2.5) and (3.7), respectively, we obtain the equation of energy of the medium, which may be presented in the form

$$\begin{aligned}\rho \frac{d}{dt}\left(U + \sum_{i=0}^{N-1} c_i \frac{v_i^2}{2}\right) = {} & \nabla_\beta (p_{i\alpha}{}^\beta v^\alpha) + (F_{M\alpha} + F_\alpha) v^\alpha + \\ & \nabla_\beta \sum_{i=0}^{N-1} (\pi_{i\alpha}{}^\beta J_i^\alpha) - \left[\tau_\alpha{}^\beta - \frac{1}{2} \sum_{i=0}^{N-1} (E_\gamma P_i^\gamma + B_\gamma M_i^\gamma)\, \delta_\alpha{}^\beta\right] \nabla_\beta v^\alpha + \\ & \sum_{i=0}^{N-1} \left[\tau_{i\alpha}{}^\beta - \frac{1}{2} (E_\gamma P_i^\gamma + B_\gamma M_i^\gamma)\, \delta_\alpha{}^\beta\right] \nabla_\beta (v^\alpha - v_i^\alpha) - \\ & \sum_{i=0}^{N-1} (F_{i\alpha} + F_{Mi\alpha} + \rho_i G_{i\alpha}) (v^\alpha - v_i^\alpha) + \\ & E_\alpha \sum_{i=0}^{N-1} \frac{d_i P_i^\alpha}{dt} + B_\alpha \sum_{i=0}^{N-1} \frac{d_i M_i^\alpha}{dt} + \rho T \frac{dS}{dt}\end{aligned} \tag{4.1}$$

where the following notation is used:

$$p_\alpha{}^\beta = \sum_{i=0}^{N-1} p_{i\alpha}{}^\beta, \quad \pi_{i\alpha}{}^\beta = \frac{1}{\rho_i}\left[p_{i\alpha}{}^\beta - \frac{1}{2} (E_\gamma P_i^\gamma + B_\gamma M_i^\gamma)\, \delta_\alpha{}^\beta\right]$$

$$\tau_\alpha{}^\beta = \sum_{i=0}^{N-1} \tau_{i\alpha}{}^\beta$$

Operator $d_i \,/\, dt$ is a derivative with respect to time determined in the system of coordinates attached to the i-th mixture constituent.

With the use of the second law of thermodynamics (2.4) it is possible to represent the energy equation (4.1) in the form

$$\begin{aligned}\rho d\left(U + \sum_{i=0}^{N-1} c_i \frac{v_i^2}{2}\right) = {} & \nabla_\beta (p_\alpha{}^\beta v^\alpha)\, dt + \left(F_{M\alpha} + \sum_{i=0}^{N-1} F_{i\alpha}\right) v^\alpha\, dt + \\ & \nabla_\beta \sum_{i=0}^{N-1} \left[\left(\pi_{i\alpha}{}^\beta - \frac{v_i^2}{2} \delta_\alpha{}^\beta\right) J_i^\alpha\right] dt + \frac{1}{2} \sum_{i=0}^{N-1} (E_\gamma P_i^\gamma + B_\gamma M_i^\gamma)\, \nabla_\alpha v_i^\alpha\, dt + \\ & \sum_{i=0}^{N-1} (F_{i\alpha} + F_{Mi\alpha}) (v_i^\alpha - v^\alpha)\, dt + \\ & E_\alpha \sum_{i=0}^{N-1} d_i P_i^\alpha + B_\alpha \sum_{i=0}^{N-1} d_i M_i^\alpha + dQ_0 + j_0^\alpha E_\alpha\, dt\end{aligned} \tag{4.2}$$

The left-hand part of this equation defines the change of total energy of the mixture considered as a single medium. The first, second, and third terms in the right-hand part of Eq. (4.2) represent the elementary work of surface (mechanical), and volume mechanical and ponderomotive forces, respectively, over elementary displacements of the medium as a whole. The fourth term represents the surface input of energy caused by diffusion processes (in particular it may contain the work of surface forces over relative displacements of constituents),

$$\frac{1}{2}\sum_{i=0}^{N-1}(E_\gamma P_i^{\gamma}+B_\gamma M_i^{\gamma})\nabla_\alpha v_i^\alpha\,dt$$

represents the elementary work of electromagnetic pressure over deformations of mixture constituents, the expression

$$\sum_{i=0}^{N-1}(F_{i\alpha}+F_{Mi\alpha})(v_i^\alpha-v^\alpha)\,dt$$

represents the elementary work of volume and ponderomotive forces over relative displacements of constituents, and the terms $E_\alpha d_i P_i^\alpha$ and $B_\alpha d_i M_i^\alpha$ are related to the elementary input of energy by the electromagnetic field to a unit of volume of the medium, caused by polarization and magnetization effects, while the last term represents the Joule heat released in a unit of volume of the medium during time dt.

In accordance with formula (2.4) the second law of thermodynamics can be represented in the form

$$\rho T\,dS=dQ^{(e)}+dQ'$$

$$dQ'=\tau^{\alpha\beta}\nabla_\beta v_\alpha\,dt+\sum_{i=0}^{N-1}\tau_i^{\alpha\beta}\nabla_\beta(v_{i\alpha}-v_\alpha)\,dt-$$

$$\sum_{i=0}^{N-1}G_{i\alpha}J_i^\alpha\,dt\qquad\left(\tau^{\alpha\beta}=\sum_{i=0}^{N-1}\tau_i^{\alpha\beta}\right)$$

where dQ' is the increment of uncompensated heat and $dQ^{(e)}$ is the input of heat, external to the medium as a whole, to a particle of the mixture and unrelated diffusion energy transfer. Assuming that the external heat input reaching the medium is due only to heat conduction and the release of Joule heat, we can write the expression for entropy increase caused by internal processes in the mixture in the form

$$\rho\,d_iS=\frac{1}{T}\sum_{i=0}^{N-1}\tau_i^{\alpha\beta}\nabla_\beta v_{i\alpha}\,dt-\frac{1}{T^2}q^\alpha\nabla_\alpha T\,dt-\frac{1}{T}\sum_{i=0}^{N-1}G_{i\alpha}J_i^\alpha\,dt+E_\alpha i_0^\alpha\,dt$$

where q^α denotes components of the heat flux vector, and allowance is made for the reversibility of the polarization and magnetization processes. The inequality

$$d_iS\geqslant 0 \tag{4.3}$$

must be satisfied for all processes.

Note that the thermodynamic forces $\tau_i^{\alpha\beta}\,/\,T$, $q^\alpha\,/\,T^2$ and $G_i^\alpha\,/\,T$ may be expressed in terms of thermodynamic fluxes $\nabla_\beta\,v_i^\alpha$, J_i^α, $\nabla_\alpha\,T$ and possibly, other determing parametrs. Homogeneous first power functions may be chosen for thermodynamic flux functions, which (in the case of symmetry of coefficents in these relationships) corresponds to Onsager formulas. In such cases inequality (4.3) must be identically satisfied. This imposes certain limitations on the selection of phenomenological coefficients.

5. On the equations of momenta for a mixture as a single medium. As the second possible system of independent variations we can take the following: - δx^α, δA_α, $\delta\varphi$, $\delta\eta_i^\alpha$, δS, $\delta_i\pi_i{}^{\hat{}\,\alpha}$, $\delta_i m_i{}^{\hat{}\,\alpha}$ which differ from the system of variations (3.1) in that instead of variations $\delta_i x^\alpha$, whose coefficients in the volume integral are equations of individual constituent momenta, variations δx^α and $\delta\eta_i^\alpha$ are considered to be arbitrary, and among the $\delta\eta_i^\alpha$ there are $3(N-1)$ independent ones. With this choice of the system of independent variables all Euler's equations (3.2) – (3.6) are valid, except Eq. (3.3) for which three equations of momenta of the mixture taken as a single medium and the $3(N-1)$ linearly independent equations obtained on the assumption of arbitrariness of variations $\delta\eta_i^\alpha$, must be substituted. Thus, instead of Eqs. (3.3) of momenta for individual constituents, we have equations of momenta for the mixture as a single medium

$$\rho a_\alpha = \nabla_\beta p_\alpha{}^\beta - \sum_{i=0}^{N-1} \nabla_\beta (v_{i\alpha} J_i{}^\alpha) + \sum_{i=0}^{N-1} F_{i\alpha} + F_{M\alpha} \tag{5.1}$$

where a_α denotes covariant components of the acceleration vector of the mixture as a single medium, which are related to components of the medium velocity vector by the equality $a_\alpha = \partial v_\alpha / \partial t + v^\beta \nabla_\beta v_\alpha$; $p_\alpha{}^\beta$ denotes mixed components of stress tensors in the mixture which are computed in the observer's system of coordinates but have the physical meaning of stress tensor components only in the system of coordinates of the mixture as a whole, and $F_{M\alpha}$ denote according to Minkowski the covariant vector components of the volume ponderomotive force exerted by the electromagnetic field on the mixture as a whole. For the stress tensor components $p_\alpha{}^\beta$ and the volume ponderomotive force we have the expressions

$$p_\alpha{}^\beta = -\rho^2 \frac{\partial U}{\partial \rho} \delta_\alpha{}^\beta + \sum_{i=0}^{N-1} \rho \frac{\partial U}{\partial \pi_i{}^\alpha} \pi_i{}^\beta + \sum_{i=0}^{N-1} \rho \frac{\partial U}{\partial m_i{}^\alpha} m_i{}^\beta +$$

$$\frac{1}{2} \sum_{i=0}^{N-1} (E_\gamma P_i{}^\gamma + B_\gamma M_i{}^\gamma) \delta_\alpha{}^\beta - \sum_{i=0}^{N-1} (E_\alpha P_i{}^\beta + B_\alpha M_i{}^\beta) + \tau_\alpha{}^\beta$$

$$F_{M\alpha} = \rho_{e0} E_\alpha + \frac{1}{c} \varepsilon_{\alpha\beta\gamma} j_0^\beta B^\gamma + \frac{1}{8\pi} (D^\gamma \nabla_\alpha E_\gamma - E_\gamma \nabla_\alpha D^\gamma + B^\gamma \nabla_\alpha H_\gamma - H_\gamma \nabla_\alpha B^\gamma)$$

where D_α and H_α in the last equality are defined by the related equalities in Sect. 3. We point out that the expression for tensor components $p_\alpha{}^\beta$ are of the form obtained above when ρ and c_i are taken as the arguments of the internal energy density function (instead of ρ_i, as was done in Sect. 3).

A comparison of equalities (3.3) and (5.1) with allowance for expressions for $p_{i\alpha}{}^\beta$, $p_\alpha{}^\beta$, and volume ponderomotive forces show that equations of momenta for the mixture taken as a single medium can be obtained from Eqs. (5.1) of momenta for individual constituents by simply summating the latter, taking into account the different method of chosing internal energy arguments.

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CHARACTER OF THE LOSS OF STABILITY IN A NONEQUILIBRIUM MAGNETIZED PLASMA

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The character of the loss of stability in a magnetized nonequilibrium plasma in a bounded region is examined. The influence of thermal conductivity and nonlinear effects are taken into account. It is shown that both magnetically soft and magnetically hard modes of the loss of stability can take place. With small parameter values of the supercritical state the rise of self-oscillations is possible. The critical value of the Hall parameter corresponding to the beginning of ionization instability [1, 2] and its dependence on the boundaries, were examined using the linear approximation in [3-5]. The spectrum for the linear problem was examined neglecting thermal conductivity [3] and taking it into account [5]. It was established that the critical value of the Hall parameter is identical in the presence of boundaries and in the case of an infinite medium. Numerical computations of the ionization instability process are given in [6, 7). The bibliography of early works on ionization instability can be found in the review [8].

1. Let us consider the behavior of a nonequilibrium magnetized plasma in a bounded region. An infinite channel extends in the direction of the y and z axes and it is bounded in the direction of the x axis by nonconducting walls, separated by a distance b. A constant magnetic field B is directed along the z axis. It is assumed that ionization equilibrium exists anf the effect of the induced magnetic field is neglected. If the electron temperature T considerably exceeds the heavy particle temperature T_a, then the state of the plasma is defined by the following system of equations:

$$\operatorname{rot}\mathbf{E}=0,\qquad \operatorname{div}\mathbf{j}=0 \tag{1.1}$$

$$I\left(\frac{\partial n}{\partial t}+\mathbf{U}\,\nabla n\right)+\frac{3}{2}nk\left(\frac{\partial T}{\partial t}+\mathbf{U}\,\nabla T\right)+\operatorname{div}\mathbf{q}=\frac{j^2}{\sigma}-\frac{3}{2}kT\delta n\nu$$

$$U=-\frac{j}{en},\quad \mathbf{j}+\mathbf{j}\times\mathbf{\Omega}=\sigma(n,T)\,\mathbf{E},\quad \mathbf{q}+\mathbf{q}\times\mathbf{\Omega}=-\lambda(n,T)\,\nabla T$$

Here I is the ionization potential, U is the directional electron velocity, σ and λ are the coefficients of electric and thermal conductivity, respectively, j is the electric current density, ν is the frequency of collisions between the electrons and heavy particles, δ is the portion of energy transferred at the collision with a heavy particle, E is the electric field strength, $\Omega=\omega/\nu(n,T)$ is the Hall parameter, ω is the electron cyclotron frequency.